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On the 'Strong-Coupling' Generalization of the Bogoliubov Model

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A generalized Bogoliubov model of the Bose gas in the ground state is proposed which properly takes into account both the long–range and short–range spatial boson correlations. It concerns equilibrium characteristics and operates with in–medium Schrödinger equations for the pair wave functions of bosons being the eigenfunctions of the second–order reduced density matrix. The approach developed provides reasonable results for a dilute Bose gas with arbitrary strong interaction between particles (the 'strong–coupling' case) and comes to the canonical Bogoliubov model in the weak–coupling regime.

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I. INTRODUCTION

It is well–known that the Bogoliubov model (BM) of the weakly interacting Bose gas [1] is a fundamental of the theory of the many–boson systems. The long–range spatial correlations of bosons are properly taken into account within this model. As to the short–range ones, there are situations when the latter need more accurate treatment. In particular, one can mention the troubles appearing within BM when arbitrary strong repulsion between bosons is expressed in a nonintegrable (singular) interparticle potential $\Phi(r)$ behaving at small separations as $1/r^m$ $(m > 3)$. These troubles are commonly overcome by means of using BM with an effective, 'dressed' interparticle potential (instead of the 'bare' one, $\Phi(r)$) that contains all the necessary 'information' concerning the short–range boson correlations [2–5].

Although there exist sufficient amount of comprehensive ways of constructing the effective interaction potentials (the pseudopotential method [2], various procedures based on summation of the ladder diagrams [3–5])[1], it looks interesting and promising to realize an alternative variant of taking account of the short–range boson correlations. We mean a generalization of BM which operates directly with the 'bare' potential $\Phi(r)$ and provides a reasonable treatment of the short–range particle correlations side by side with the long–range ones. A generalization of BM like this is proposed in the present Letter. Note that we limit ourselves to the case of the zero temperature and consider only equilibrium characteristics such as the pair correlation function and boson momentum distribution.

The key point of generalizing BM is based on rejecting the usual way of dealing with the Bogoliubov model. The investigation presented concerns the second–order reduced density matrix (2-matrix). In particular, we operate with in–medium Schrödinger equations whose solutions are the eigenfunctions of the 2–matrix, or the pair wave functions. As an in–medium interparticle potential depends on these functions, so the cited Schrödinger equations are nonlinear ones. However, they can be linearized in the weak–coupling regime as well as for a dilute Bose gas even with strong repulsive interaction between bosons. The former case corresponds to the canonical Bogoliubov model. The latter variant is related to, say, the 'strong–coupling' generalization of BM.

The present Letter is organized as follows. In the second part BM is reconsidered in the framework of the 2–matrix. The third section concerns the pair wave functions in the generalized BM. At last, to show a reasonable character of the approach proposed, the zero–density limit for the Bose gas with strong repulsion between bosons is discussed in the fourth part of the paper.

II. THE BOGOLIUBOV MODEL IN THE LIGHT OF THE 2–MATRIX

Let us consider a homogeneous cold many–body system of N spinless bosons with the volume V and interparticle potential $\Phi(r)$. Note that absence of the spin degrees of freedom simplifies the further reasoning without a loss of generality.

[1]All the procedures can be reduced to the ordinary two–particle Schrödinger equation with the 'bare' potential.

To start our investigation, let us recall the necessary definitions. The 2–matrix for the system of interest is written as follows:

$$\rho_2(\mathbf{r}'_1,\mathbf{r}'_2;\mathbf{r}_1,\mathbf{r}_2)=\sum_\nu w_\nu\,\psi_\nu(\mathbf{r}'_1,\mathbf{r}'_2)\,\psi^*_\nu(\mathbf{r}_1,\mathbf{r}_2)\,, \qquad (1)$$

where $\psi_\nu(\mathbf{r}_1,\mathbf{r}_2)$ are usually called [6] the pair wave functions and, physically, $\sum_\nu w_\nu = 1$ $(w_\nu \geq 0)$. The pair wave functions for bosons are symmetric with respect to the permutation of particles and obey the standard normalization condition

$$\int_V \int_V |\psi_\nu(\mathbf{r}_1,\mathbf{r}_2)|^2\, d^3r_1\, d^3r_2 = 1.$$

The 2–matrix is connected with the pair correlation function

$$F_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}'_1,\mathbf{r}'_2)=\langle\psi^+(\mathbf{r}_1)\psi^+(\mathbf{r}_2)\psi(\mathbf{r}'_2)\psi(\mathbf{r}'_1)\rangle \qquad (2)$$

by the expression [6]

$$\rho_2(\mathbf{r}'_1,\mathbf{r}'_2;\mathbf{r}_1,\mathbf{r}_2)=\frac{F_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}'_1,\mathbf{r}'_2)}{N(N-1)}\,. \qquad (3)$$

Here $\psi(\mathbf{r}_1)$ denotes the boson field operator. Knowing the pair correlation function, one is able to calculate all the important thermodynamic quantities [7].

The most general structure of the 2-matrix of the equilibrium many–body system of spinless bosons is given by the following expression [8,9]:

$$F_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}'_1,\mathbf{r}'_2)=\sum_{\omega,\mathbf{q}}\frac{N_{\omega,q}}{V}\varphi^*_{\omega,q}(\mathbf{r})\varphi_{\omega,q}(\mathbf{r}')$$

$$\times\exp\{i\mathbf{q}(\mathbf{R}'-\mathbf{R})\}$$

$$+\sum_{\mathbf{p},\mathbf{q}}\frac{N_{\mathbf{p},\mathbf{q}}}{V^2}\varphi^*_{\mathbf{p},\mathbf{q}}(\mathbf{r})\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r}')\exp\{i\mathbf{q}(\mathbf{R}'-\mathbf{R})\}, \qquad (4)$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$. The quantity $\varphi_{\omega,q}(\mathbf{r})\cdot\exp(i\mathbf{q}\mathbf{R})/\sqrt{V}$ denotes the wave function of the $\omega-$th bound state of the pair of bosons with the total momentum $\hbar\mathbf{q}$. Respectively, $\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r})\cdot\exp(i\mathbf{q}\mathbf{R})/V$ stands for the wave function of a dissociated state of the pair of bosons with the total momentum $\hbar\mathbf{q}$ and the momentum of relative motion $\hbar\mathbf{p}$. For the characteristics $N_{\omega,q}$ and $N_{\mathbf{p},\mathbf{q}}$ we have: $N_{\omega,q}$ is the duplicated number of the bound pairs of the $\omega-$th species with the total momentum $\hbar\mathbf{q}$; $N_{\mathbf{p},\mathbf{q}}$ is the duplicated number of the dissociated pairs with the total momentum $\hbar\mathbf{q}$ and the momentum of relative motion $\hbar\mathbf{p}$. The wave functions $\varphi_{\omega,q}(\mathbf{r})$ and $\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r})$ obey the normalization conditions

$$\lim_{V\to\infty}\int_V |\varphi_{\omega,q}(\mathbf{r})|^2\, d^3r = 1,$$

$$\lim_{V\to\infty}\frac{1}{V}\int_V |\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r})|^2\, d^3r = 1, \qquad (5)$$

and have the symmetry properties

$$\varphi_{\omega,q}(\mathbf{r})=\varphi_{\omega,q}(-\mathbf{r})\,,$$

$$\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r})=\varphi_{\mathbf{p},\mathbf{q}}(-\mathbf{r})=\varphi_{-\mathbf{p},\mathbf{q}}(\mathbf{r})\,,$$

which are a consequence of the Bose statistics. Thus, the first term in the right–hand side of (4) represents the sector of the bound pairs; the second one corresponds to the dissociated states. Remark that generally speaking, one can expect a discrete index to appear in addition to $\mathbf{q}$ and $\mathbf{p}$ for the dissociated states in rather complicated situations. However, this does not concern our present consideration. So, we have restricted ourselves to the summation over $\mathbf{q}$ and $\mathbf{p}$ in the second term of the right–hand side of (4).

Comprehensive analysis recently fulfilled in paper [9], has demonstrated that, in the thermodynamic limit, the correlation function (4) can be rewritten as

$$F_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r'}_1,\mathbf{r'}_2)=n_0^2\varphi(r)\varphi(r')+\int d^3p\,d^3q\,\frac{n_0}{(2\pi)^3}$$

$$\times\left\{\delta\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)n\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)+\delta\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)n\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)\right\}$$

$$\times\,\varphi^*_{\mathbf{p}}(\mathbf{r})\varphi_{\mathbf{p}}(\mathbf{r}')\exp\{i\mathbf{q}(\mathbf{R}'-\mathbf{R})\}+\widetilde{F}_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r'}_1,\mathbf{r'}_2), \qquad (6)$$

where n_0 denotes the density of the condensed particles; $n(p)=n(\mathbf{p})$ stands for the distribution of the noncondensed bosons over momenta. Note that the Bose–Einstein condensation of particles is accompanied by the condensation of the particle pairs and, thus, by the appearance of the $\delta-$functional terms (the off–diagonal long–range order) in the pair distribution over momenta $\hbar\,\mathbf{p}$ and $\hbar\,\mathbf{q}$ [10]. The first term in the right–hand side of (6) is conditioned by presence of a macroscopic number of the pairs with $q=p=0$. Since they include only the condensed bosons, we can call them the condensate–condensate pairs. The second term in (6) corresponds to the condensate–supracondensate couples. Besides a condensed particle, they also include a noncondensed boson. At last, $\widetilde{F}_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}'_1,\mathbf{r}'_2)$ is the contribution made by the supracondensate–supracondensate dissociated states of a pair and, maybe, by its bound states. For the wave functions of the condensate–condensate and condensate–supracondencate couples we have

$$\varphi(r)=1+\psi(r),\;\varphi_{\mathbf{p}}(\mathbf{r})=\sqrt{2}\cos(\mathbf{pr})+\psi_{\mathbf{p}}(\mathbf{r})\quad(p\neq 0), \qquad (7)$$

where the boundary conditions

$$\psi(r) \to 0 \ \ (r \to \infty), \quad \psi_{\mathbf{p}}(\mathbf{r}) \to 0 \ \ (r \to \infty) \tag{8}$$

take place. At small particle separations the pair wave function $\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r})$ is very close to the usual wave function of the two–body problem with the relative momemtum p. Therefore, for a singular interparticle potential, when $\Phi(r) \propto 1/r^m$ $(m > 3)$ at small r, we have $\varphi_{\mathbf{p},\mathbf{q}}(\mathbf{r}) \to 0$ as $r \to 0$. And, hence, the relations

$$\psi(r=0) = -1, \quad \psi_{\mathbf{p}}(\mathbf{r}=0) = -\sqrt{2} \tag{9}$$

are fulfilled.

In the case of a small depletion of the zero momentum state (it is of interest in this Letter) we can neglect the third term in expression (6):

$$F_2(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}'_1, \mathbf{r}'_2) = n_0^2\, \varphi^*(r)\, \varphi(r')$$
$$+ \frac{16 n_0}{(2\pi)^3} \int d^3p\, n(2p) \varphi^*_{\mathbf{p}}(\mathbf{r}) \varphi_{\mathbf{p}}(\mathbf{r}') \exp\{i2\mathbf{p}(\mathbf{R}' - \mathbf{R})\}. \tag{10}$$

As it is known, there are two physical situations when the Bose condensate fraction is expected to be close to 1. One of them is related to the weak–coupling regime when a small depletion of the zero momentum state results from a weak interaction of bosons. The Bogoliubov model is an adequate approach of investigating this case. Another situation occurs when we deal with a dilute Bose gas with an arbitrary strong interaction between particles (singular potential). Here the dilution of the system gives rise to the small depletion. In this 'strong–coupling' regime the short–range correlations play a significant role, which is expressed in relations (9). On the contrary, the weak–coupling case is specified by the inequalities

$$|\psi(r)| \ll 1, \quad |\psi_{\mathbf{p}}(\mathbf{r})| \ll 1\,. \tag{11}$$

In particular, the Bogoliubov model is characterized by the choice [9]

$$|\psi(r)| \ll 1, \quad \psi_{\mathbf{p}}(\mathbf{r}) = 0\,. \tag{12}$$

Expressions (10) and (12) allow one to obtain

$$F_2(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1, \mathbf{r}_2) = n^2 g(r) = n_0^2 \Big(1 + \psi(r) + \psi^*(r)\Big)$$
$$+ 2n_0 \left(n - n_0 + \frac{1}{(2\pi)^3} \int n(k) \exp(i\mathbf{k}\mathbf{r})\, d^3k \right), \tag{13}$$

where $n = N/V$ and $g(r)$ is the radial distribution function. According to the weak–coupling conditions (11) and the approximation adopted in (10), it is correct to neglect the terms of the order of $\psi(r)(n - n_0)$ and $(n - n_0)^2$ in (13). Besides, we may choose the wave function of the pair ground state as a real quantity [11]: $\psi(r) = \psi^*(r)$. So, expression (13) can be rewritten as

$$g(r) = 1 + 2\psi(r) + \frac{2}{(2\pi)^3 n} \int n(k) \exp(i\mathbf{k}\mathbf{r})\, d^3k. \tag{14}$$

Let us show that (14) does represent the result of BM. To be convinced of this, we need the equality

$$\widetilde{\psi}(k) = \int \psi(r) \exp(-i\mathbf{k}\mathbf{r})\, d^3r = \frac{1}{n_0} \langle a_{\mathbf{k}}\, a_{-\mathbf{k}} \rangle \tag{15}$$

connecting $\psi(r)$ with the boson annihilation operators [9]. At T=0 (we deal with the zero temperature case in the present Letter) BM yields [1,6] the following relations:

$$\langle a_{\mathbf{k}}\, a_{-\mathbf{k}} \rangle = \frac{A_k}{1 - A_k^2}, \quad n(k) = \frac{A_k^2}{1 - A_k^2}\,, \tag{16}$$

where

$$A_k = \frac{1}{n_0 \widetilde{\Phi}(k)} \left(E(k) - \frac{\hbar^2 k^2}{2m} - n_0\, \widetilde{\Phi}(k) \right) \tag{17}$$

and

$$E(k) = \sqrt{\frac{\hbar^2 k^2}{m}\, n_0 \widetilde{\Phi}(k) + \frac{\hbar^4 k^4}{4m^2}},$$
$$\widetilde{\Phi}(k) = \int \Phi(r) \exp(-i\mathbf{k}\mathbf{r})\, d^3r. \tag{18}$$

Using (14), (15) and (16) one is able to arrive at

$$g(r) = 1 + \frac{2}{(2\pi)^3\, n} \int \frac{A_k}{1 - A_k}\, \exp(i\mathbf{k}\mathbf{r})\, d^3k\,. \tag{19}$$

This relation is exactly the result of the Bogoliubov model (see Ref. [6]).

Concluding this part of the paper, let us take notice of an interesting equation following from (15) – (18) and being important for the reasoning of the next section. It is given by

$$-\frac{\hbar^2 k^2}{m} \widetilde{\psi}(k) = \widetilde{\Phi}(k) + 2\, \widetilde{\Phi}(k) \left(n(k) + n_0\, \widetilde{\psi}(k) \right), \tag{20}$$

where $n_0\, \widetilde{\psi}(k)$ can be replaced by $n\, \widetilde{\psi}(k)$ because we agreed to neglect the terms of the order of $(n - n_0)\, \psi(r)$. From (14) and (20) it follows that

$$\frac{\hbar^2}{m} \nabla^2 \varphi(r) = \Phi(r) + n \int \Phi(|\mathbf{r} - \mathbf{y}|) \Big(g(y) - 1\Big)\, d^3y\,. \tag{21}$$

This looks like the Schrödinger equation in the Born approximation.

III. PAIR WAVE FUNCTIONS IN THE GENERALIZED BOGOLIUBOV MODEL

As it has been noted above, this Letter addresses the generalization of the Bogoliubov model in such a way that the short–range correlations should be taken into account properly side by side with the long–range ones, a small depletion of the zero momentum state being implied while generalizing. So, in our further investigation it is correct to rely on expression (10) taken beyond the weak–coupling regime introduced by (11).

To employ approximation (10) for the 2-matrix, one needs to determine the wave functions $\varphi(r)$ and $\varphi_{\mathbf{p}}(\mathbf{r})$ beyond the weak coupling. To do this, let us consider the in–medium two–particle problem:

$$H_{12}\ \psi_\nu(\mathbf{r}_1,\mathbf{r}_2) = E_\nu\ \psi_\nu(\mathbf{r}_1,\mathbf{r}_2)\ . \tag{22}$$

The Hamiltonian H_{12} of two bosons placed into the medium of similar bosons can be represented as

$$H_{12} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + \Phi(|\mathbf{r}_1-\mathbf{r}_2|) + U_1 + U_2\ , \tag{23}$$

where U_i $(i=1,2)$ stands for the energy of the interaction of the $i-$th particle with the medium. Proceeding in the spirit of the Thomas–Fermi approach (for details see Ref. [12]) and, thus, neglecting retarding effects, one is able to approximate U_i in the form

$$U_i = (N-2)\int \Phi(|\mathbf{r}_i-\mathbf{r}_3|) w(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\, d^3r_3 \quad (i=1,2), \tag{24}$$

where $w(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ denotes the density of the probability of observing the third particle at the point $\mathbf{r}_3$ under the condition that the first and second ones are located at $\mathbf{r}_1$ and $\mathbf{r}_2$. This quantity is connected with the third and second reduced density matrices via the relation

$$w(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \rho_2^{-1}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1,\mathbf{r}_2)\,\rho_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\,. \tag{25}$$

Using the Kirkwood superposition approximation [13]

$$V^3\ \rho_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \simeq g(|\mathbf{r}_1-\mathbf{r}_2|)\, g(|\mathbf{r}_1-\mathbf{r}_3|)\, g(|\mathbf{r}_2-\mathbf{r}_3|)\ , \tag{26}$$

one can obtain

$$w(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \simeq \frac{1}{V}\, g(|\mathbf{r}_1-\mathbf{r}_3|)\, g(|\mathbf{r}_2-\mathbf{r}_3|)\ . \tag{27}$$

With (24) and (27) we arrive at

$$U_1 = U_2 = n\int g(|\mathbf{r}-\mathbf{y}|)\,\Phi(|\mathbf{r}-\mathbf{y}|)\ g(y)\ d^3y\ . \tag{28}$$

Thus, equation (22) taken with the specifications (23) and (28) separates in the usual variables $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$,

$$\psi_\nu(\mathbf{r}_1,\mathbf{r}_2) = \varphi_{\mathbf{p}}(\mathbf{r})\,\frac{\exp(i\mathbf{q}\mathbf{R})}{\sqrt{V}}\ ,$$

and yields the following relation for the wave function $\varphi_{\mathbf{p}}(\mathbf{r})$:

$$-\frac{\hbar^2}{m}\nabla^2\,\varphi_{\mathbf{p}}(\mathbf{r}) + \Phi(r)\varphi_{\mathbf{p}}(\mathbf{r}) + 2\varphi_{\mathbf{p}}(\mathbf{r})n\int g(|\mathbf{r}-\mathbf{y}|)\Phi(|\mathbf{r}-\mathbf{y}|)g(y)\, d^3y = \varepsilon_p\varphi_{\mathbf{p}}(\mathbf{r}). \tag{29}$$

Let us consider equation (29). The quantity ε_p can be found taking the limit $r\to\infty$. Using the asymptotic relations

$$\lim_{r\to\infty} g(r) = 1\ ,\quad \lim_{r\to\infty}\Phi(r) = 0\ ,$$

we readily obtain at $r\to\infty$

$$\int g(|\mathbf{r}-\mathbf{y}|)\Phi(|\mathbf{r}-\mathbf{y}|)g(y)\, d^3y \to \int g(|\mathbf{r}-\mathbf{y}|)\Phi(|\mathbf{r}-\mathbf{y}|)\, d^3y.$$

Thus, we arrive at

$$\varepsilon_p = \frac{\hbar^2 p^2}{m} + 2n\int g(|\mathbf{r}-\mathbf{y}|)\,\Phi(|\mathbf{r}-\mathbf{y}|)\ d^3y \tag{30}$$

rather than $\varepsilon_p = \hbar^2 p^2/m$ which appears within the ordinary two–body problem. Inserting (30) into (29), one is able to find

$$\frac{\hbar^2}{m}\nabla^2\varphi_{\mathbf{p}}(\mathbf{r}) = -\frac{\hbar^2p^2}{m} + \Phi(r)\varphi_{\mathbf{p}}(\mathbf{r}) + 2\xi_{ex}n\varphi_{\mathbf{p}}(\mathbf{r})\int g(|\mathbf{r}-\mathbf{y}|)\Phi(|\mathbf{r}-\mathbf{y}|)\Big(g(y)-1\Big)\, d^3y, \tag{31}$$

where the equality

$$\int g(|\mathbf{x}-\mathbf{y}|)\,\Phi(|\mathbf{x}-\mathbf{y}|)\ d^3y \ = \ \int g(|\mathbf{z}-\mathbf{y}|)\,\Phi(|\mathbf{z}-\mathbf{y}|)\ d^3y$$

is used. In (31) ξ_{ex} is the correcting factor which should be introduced to compensate oversimplification of treating the exchange effects while deriving (29). Indeed, the exchange between bosons in the pair is taken into consideration: $\varphi_{\mathbf{p}}(\mathbf{r}) = \varphi_{\mathbf{p}}(-\mathbf{r}) = \varphi_{-\mathbf{p}}(\mathbf{r})$. However, the arguments resulting in (29) ignore the exchange between the particles of the pair and surrounding bosons whose influence is considered on the mean–field level, in the spirit of the Thomas–Fermi approach. This correcting

factor may be, in general, a function of $\mathbf{p}$ and some other quantities related to the problem: $\xi_{ex} = \xi_{ex}(\mathbf{p}, \ldots)$. For $\mathbf{p} = 0$ equation (31) has to come to (21) provided relations (12) are valid, which allows one to find in the weak–coupling regime $\xi_{ex}(\mathbf{p} = 0) = 1/2$. In this case the approximation $\xi_{ex} = 1/2$ can also be employed for $\mathbf{p} \neq 0$ owing to a small mean–absolute value of a boson momentum. Moreover, we expect that in the situation of the large condensate fraction, the choice $\xi_{ex} = 1/2$ is correct beyond the weak coupling too. The main reason for this is that the interaction between a couple of particles and the medium is weak in both the cases. For the potentials with a repulsive core this is due to a small density of surrounding bosons.

Remark, that in contrast to (21), relation (31) is reduced to the usual Schrödinger equation of the two–body problem in the limit $n \to 0$. The same occurs for $r \to 0$ in the situation of arbitrary strong repulsion between bosons. Indeed, in this case $\Phi(r) \to \infty$ at $r \to 0$. Hence, the second term in the right–hand side of (31) becomes much less than $\Phi(r)$ at small r. This leads to conditions (9) fulfilled at any particle density. It is noteworthy that (31) with $\xi_{ex} = 1/2$ coincides with one of the basic relations of the approach developed in papers [14].

An important peculiarity of equation (31) is that it can be used without any divergency in the integral term in the case of a singular interparticle potential because

$$g(r)\,\Phi(r) \to 0 \quad (r \to 0)\,.$$

Thus, (31) reduced to (21) in the weak–coupling approximation, well answer our purpose of generalizing BM.

IV. 'STRONG–COUPLING' CASE

To deal with the generalization of BM based on (10) and (31) with $\xi_{ex} = 1/2$, we need one more equation. This is because the number of the functions $g(r)$, $n(k)$, $\varphi(r)$, $\varphi_{\mathbf{p}}(\mathbf{r})$ to be determined is larger than the number of the equations at our disposal. Within the Bogoliubov model for a weakly interacting Bose gas, the relation additional to (14) and (21) at zero temperature is of the form

$$n^2\, \widetilde{\psi}^2(k) = n(k)\, \left(n(k)+1\right), \tag{32}$$

(see relations (15) – (18)). Remark that (32) follows from the canonical character of the well–known Bogoliubov transformation [1,6]. The question now arises if one may employ (32) beyond the weak coupling or not. It turns out that (32) yields quite reasonable results even in the case of a dilute Bose gas with strong repulsive interaction between bosons. To be convinced of this, let us consider equations (10), (31) and (32) in the 'strong–coupling' regime. From (32) it follows that

$$n(k) = \frac{1}{2}\left(\sqrt{1 + 4\,n^2\,\widetilde{\psi}^2(k)} - 1\right)\,.$$

Therefore, in the limit $n \to 0$ we arrive at

$$\frac{n(k)}{n} = n\, \widetilde{\psi}_0^2(k)\,, \tag{33}$$

where $\psi_0(r)$ obeys equation (31) taken at $n = 0$ and $p = 0$:

$$\frac{\hbar^2}{m}\nabla^2\Big(1+\psi_0(r)\Big) = \Big(1+\psi_0(r)\Big)\,\Phi(r)\,. \tag{34}$$

The relation (33) suggests that all the bosons are condensed in the zero–density limit. So, the use of (32) does not contradict the common expectation concerning a large condensate fraction in a dilute Bose gas with strong repulsive interaction. According to equation (31), for sufficiently low values of p we have

$$\varphi_{\mathbf{p}}(\mathbf{r}) \simeq \sqrt{2}\,\varphi(r)\ \cos(\mathbf{pr})\,. \tag{35}$$

Since (33) is valid at small boson densities n, one can take approximation (35) to investigate the thermodynamics of a dilute Bose gas. Inserting (35) into (10) we obtain the following ansatz:

$$g(r) = \varphi^2(r)\Big(1 + \frac{2}{(2\pi)^3\,n}\int\, n(k)\exp(i\mathbf{kr})\,d^3k\Big)\,, \tag{36}$$

where $\varphi(r)$ is given by equation (31) at $\mathbf{p} = 0$ and $\xi_{ex} = 1/2$:

$$\frac{\hbar^2}{m}\nabla^2\varphi(r) = \Phi(r)\varphi(r)$$

$$+\,n\,\varphi(r)\int g(|\mathbf{r}-\mathbf{y}|)\Phi(|\mathbf{r}-\mathbf{y}|)\Big(g(y)-1\Big)\,d^3y. \tag{37}$$

This ansatz can be used for arbitrary strong repulsion between bosons without any divergency because $\varphi(r)\,\Phi(r) \to 0$ at $r \to 0$ while $\Phi(r) \to \infty$. It is worth noting that ansatz (36) is also good for a weakly interacting Bose gas. Really, (36) is reduced to (14) with the assumption $|\psi(r)| \ll 1$ and neglect of the term containing the product $\psi(r)\,n(k)$. At last, using (36) and (37) and taking the zero density limit, one can derive

$$\lim_{n\to 0}\ g(r) = \varphi_0^2(r)\,, \tag{38}$$

where $\varphi_0(r) = 1+\psi_0(r)$. Equality (38) is the well–known result for the pair distribution function of the Bose gas of strongly interacting particles [1].

V. CONCLUSION

The generalization of the Bogoliubov model of the cold Bose gas has been proposed which is based on equations (32), (36) and (37). They come from the more complicated set of equations (10), (31) with $\xi_{ex} = 1/2$ and (32) provided the ansatz (35) is used. The generalization properly takes into account the short–range boson correlations side by side with the long–range ones. The proposed approach yields reasonable results in the weak–coupling regime as well as in the 'strong–coupling' case of a dilute Bose gas with arbitrary intense repulsion. The detailed analysis of the latter variant will be fulfilled in the forthcoming paper. As it has been noted in the Introduction, the in–medium Schrödinger equations can be linearized not only in the weak–coupling approximation. This can also be done in the 'strong–coupling' regime. In the former case (37) is reduced to equation (21) linear in $\psi(r)$. While in the latter situation (37) comes to an equation linear in $\zeta(r) = \varphi(r) - \varphi_0(r)$ due to the obvious inequality $|\zeta(r)| \ll |\varphi_0(r)| (n \to 0)$.

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